Theorem:- The Diophantine equation $x^4+y^4=z^2$ has no solution in positive integers x, y, z.

Proof:- We will prove it by contradiction. So assume that x_0, y_0, z_0 be a solution to $x^4+y^4=z^2$.

Without any lost we can assume $\gcd(x_0, y_0)=1$.

If not suppose $\gcd(x_0, y_0)=d$. then $\exists\ x_1, y_1 \in \mathbb{Z}$ such that $x_0 = dx_1$, $y_0 = dy_1$ and $\gcd(x_1, y_1)=1$

Now $x_0^4 + y_0^4 = z_0^2$

$\Rightarrow d^4x_1^4 + d^4y_1^4 = z_0^2$. so $d^4 \mid z_0^2 \Rightarrow z_0^2 = d^4 z_1^2$

$\Rightarrow z_0 = d^2 z_1$

$\Rightarrow d^4x_1^4 + d^4y_1^4 = d^4 z_1^2$

$\Rightarrow x_1^4 + y_1^4 = z_1^2$ show that x_1, y_1, z_1 is a solution.

Now $x_0^4 + y_0^4 = z_0^2 \Rightarrow (x_0^2)^2 + (y_0^2)^2 = z_0^2$

So by solution of Pythagorean equation we have integers s, t, satisfying $s > t > 0$ s.t

$$x_0^2 = 2st, \quad y_0^2 = s^2 - t^2 \quad z_0 = s^2 + t^2$$

where exactly one of s & t is even,

If s is even then $s^2 \equiv 0 \pmod 4$ and t odd

$\Rightarrow t^2 \equiv 1 \pmod 4$

then $y_0^2 = s^2 - t^2 \equiv 0 - 1 \equiv -1 \pmod 4$

i.e. $y_0^2 \equiv 3 \pmod 4$ is a contradiction.

So s is odd and t is even.

Let $t = 2r$ then

$$x_0^2 = 2s \cdot 2r = 4sr \Rightarrow \left(\frac{x_0}{2}\right)^2 = sr$$

Now $\gcd(s,t) = 1 \Rightarrow \gcd(s,r) = 1.$

Since product of two relatively prime integers is a square so each s & r are a perfect square.

So $\exists$ u_1 and v_1 such that

$$s = u_1^2 \quad \& \quad r = v_1^2$$

Now from $y_0^2 = s^2 - t^2 \Rightarrow t^2 + y_0^2 = s^2$

Since $\gcd(s,t) = 1$ therefore $\gcd(t, y_0, s) = 1$ i.e. t, y_0, s a primitive Pythagorean triple.

So there exists the integers u, v, with $u > v$.

such that
$$t = 2uv$$
$$y_0 = u^2 - v^2$$
$$s = u^2 + v^2$$

Now $uv = \frac{t}{2} = r = v_1^2$

So $\exists$ $x_1, y_1 \in$ such that $u = x_1^2$, & $v = y_1^2$

Then $s = u^2 + v^2 \Rightarrow x_1^4 + y_1^4 = u_1^2$.

Putting $z_1 = u_1$ we get ~~a~~ another solution x_1, y_1, z_1 such that

$$0 < z_1 < z_1^2 = s \leq s^2 < s^2 + t^2 = z_0$$

i.e. $0 < z_1 < z_0$

Similarly from x_1, y_1, z_1 we can another set of primitive solution x_2, y_2, z_2 such that

$$z_0 > z_1 > z_2$$

Thus we get a ~~degree~~ decreasing infinite sequence of positive integers $z_0 > z_1 > z_2 > z_3 \cdots\cdots$

which is a contradiction as this sequence must have a smallest integer.

This contradiction shows that there does not exist any positive integral solution of

$$x^4 + y^4 = z^2$$

<u>Cor:-</u> The equation $x^4 + y^4 = z^4$ has no soln.

<u>Proof:-</u> If x_0, y_0, z_0 be a soln of the eqn then x_0, y_0, z_0^2 is a soln of $x^4 + y^4 = z^2$, which contradicts the above theorem. Hence the cor.

PROBLEMS 12.2

1. Show that the equation $x^2 + y^2 = z^3$ has infinitely many solutions for x, y, z positive integers.
[*Hint:* For any $n \geq 2$, let $x = n(n^2 - 3)$ and $y = 3n^2 - 1$.]
2. Prove the theorem: The only solutions in nonnegative integers of the equation $x^2 + 2y^2 = z^2$, with $\gcd(x, y, z) = 1$, are given by
$$x = \pm(2s^2 - t^2) \qquad y = 2st \qquad z = 2s^2 + t^2$$
where s, t are arbitrary nonnegative integers.
[*Hint:* If u, v, w are such that $y = 2w$, $z + x = 2u$, $z - x = 2v$, then the equation becomes $2w^2 = uv$.]
3. In a Pythagorean triple x, y, z, prove that not more than one of x, y, or z can be a perfect square.
4. Prove each of the following assertions:
(a) The system of simultaneous equations
$$x^2 + y^2 = z^2 - 1 \qquad \text{and} \qquad x^2 - y^2 = w^2 - 1$$
has infinitely many solutions in positive integers x, y, z, w.
[*Hint:* For any integer $n \geq 1$, take $x = 2n^2$ and $y = 2n$.]
(b) The system of simultaneous equations
$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x^2 - y^2 = w^2$$
admits no solution in positive integers x, y, z, w.
(c) The system of simultaneous equations
$$x^2 + y^2 = z^2 + 1 \qquad \text{and} \qquad x^2 - y^2 = w^2 + 1$$
has infinitely many solutions in positive integers x, y, z, w.
[*Hint:* For any integer $n \geq 1$, take $x = 8n^4 + 1$ and $y = 8n^3$.]
5. Use Problem 4 to establish that there is no solution in positive integers of the simultaneous equations
$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x^2 + 2y^2 = w^2$$
[*Hint:* Any solution of the given system also satisfies $z^2 + y^2 = w^2$ and $z^2 - y^2 = x^2$.]
6. Show that there is no solution in positive integers of the simultaneous equations
$$x^2 + y^2 = z^2 \qquad \text{and} \qquad x^2 + z^2 = w^2$$
hence, there exists no Pythagorean triangle whose hypotenuse and one of whose sides form the sides of another Pythagorean triangle.
[*Hint:* Any solution of the given system also satisfies $x^4 + (wy)^2 = z^4$.]
7. Prove that the equation $x^4 - y^4 = 2z^2$ has no solutions in positive integers x, y, z.
[*Hint:* Because x, y must be both odd or both even, $x^2 + y^2 = 2a^2$, $x + y = 2b^2$, $x - y = 2c^2$ for some a, b, c; hence, $a^2 = b^4 + c^4$.]
8. Verify that the only solution in relatively prime positive integers of the equation $x^4 + y^4 = 2z^2$ is $x = y = z = 1$.
[*Hint:* Any solution of the given equation also satisfies the equation
$$z^4 - (xy)^4 = \left(\frac{x^4 - y^4}{2}\right)^2.]$$

9. Prove that the Diophantine equation $x^4 - 4y^4 = z^2$ has no solution in positive integers x, y, z.
[*Hint:* Rewrite the given equation as $(2y^2)^2 + z^2 = (x^2)^2$ and appeal to Theorem 12.1.]
10. Use Problem 9 to prove that there exists no Pythagorean triangle whose area is twice a perfect square.
[*Hint:* Assume to the contrary that $x^2 + y^2 = z^2$ and $\frac{1}{2}xy = 2w^2$. Then $(x+y)^2 = z^2 + 8w^2$, and $(x-y)^2 = z^2 - 8w^2$. This leads to $z^4 - 4(2w)^4 = (x^2 - y^2)^2$.]
11. Prove the theorem: The only solutions in positive integers of the equation

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{z^2} \qquad \gcd(x, y, z) = 1$$

are given by

$$x = 2st(s^2 + t^2) \qquad y = s^4 - t^4 \qquad z = 2st(s^2 - t^2)$$

where s, t are relatively prime positive integers, one of which is even, with $s > t$.
12. Show that the equation $1/x^4 + 1/y^4 = 1/z^2$ has no solution in positive integers.