# Permutations

Brajen Das

**Def:** Let $\Omega = \{1, 2, \ldots, n\}$ be a finite subset of the set of natural numbers. Then a one-one onto function from $\Omega$ to $\Omega$ is called a permutation on $\Omega$.

For example: $\Omega = \{1, 2, 3, 4, 5\}$

Then $f: \Omega \to \Omega$ s.t. $f(1) = 3$, $f(2) = 1$, $f(3) = 5$, $f(4) = 2$, $f(5) = 4$ is a permutation.

We represent this permutation as follows.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$$

The first row is the pre-images and the second row is the images of the first row.

We can change the position of the columns.

i.e. $$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 & 5 & 4 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}.$$

In general.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ f(1) & f(2) & f(3) & f(4) & \cdots & f(n) \end{pmatrix}$$

For a fixed $\Omega$, each permutation is a one-one function from $\Omega$ to $\Omega$.

So the permutations can be composed.

Let us recall composition of functions.

Let $f: A \to B$ and $g: B \to C$ be two functions. Then the composition of $f$ and $g$ denoted by $(g \circ f)$ is a function from $A \to C$ is defined by $(g \circ f)(x) = g(f(x)) \quad \forall x \in A$.

Similarly If $f$ and $g$ be two permutations on $\Omega$ then $f: \Omega \to \Omega$ and $g: \Omega \to \Omega$ be two 1-1 onto functions. So we can find both $g \circ f$ and $f \circ g$. $g \circ f$ will be denoted by $gf$ and $f \circ g$ will be denoted by $fg$.

For example $\Omega = \{1, 2, 3, 4, 5\}$

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

Then $gf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$

$(g \circ f)(1) = g(f(1)) = g(2) = 1$

$(g \circ f)(2) = g(f(2)) = g(5) = 4$

$(g \circ f)(3) = g(f(3)) = g(3) = 2$

and similarly $fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix}$

Note that $gf \neq fg$

Since $f$ and $g$ both are 1-1 onto functions from $\Omega$ to $\Omega$ so $gf$ and $fg$ are also 1-1 onto fn from $\Omega$ to $\Omega$. i.e. $gf$ and $fg$ are also permutations on $\Omega$.

Thus compositions of permutations is a binary operation.

Theorem: Let $\Omega = \{1, 2, \cdots n\}$ and $S_n$ be the set of all permutations. Then $S_n$ is a group under composition of permutations.

Proof: (i) Let $f$ & $g \in S_n$, then $f$ & $g$ both are 1-1 onto functions on $\Omega$. So $gf = g \circ f$ and $fg = f \circ g$ both are one-one onto functions on $\Omega$. So $gf$ and $fg$ are permutations. Hence the composition of permutation is a binary operation on $S_n$

(ii) Since composition of functions is associative therefore composition of permutations is also associative

(iii) Let $I$ be the indentity permutation. i.e. $I = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$. Then for any $f \in S_n$
$If = I \circ f = f$ and $fI = f \circ I = f$. i.e. $I$ is the identity with respect to composition of permutation.

(iv) Let $f \in S_n$ then $f$ is a one-one onto function on $\Omega$. So $f^{-1}$ exist and is also a 1-1 onto function on $\Omega$. So $f^{-1} \in S_n$ and

$$ff^{-1} = f \circ f^{-1} = I \quad \text{and} \quad f^{-1}f = f^{-1} \circ f = I$$

ie $f^{-1}$ is the inverse of $f$

That is each permutation has its inverse in $S_n$

Hence $S_n$ is a group under composition of permutations.

We have already seen that for ~~an~~ arbitrary $g, f \in S_n$ $gf \neq fg$

Thus $S_n$ is a non abelian group under composition of permutations.

Cycle:- Instead of writing a permutation in two rows we can express it in one row as follows—

We start with an element followed by its image, then image of second entry and so on. The image of last entry will be the first element. This type of representation is called a cycle. It should be noted that, if the image of an element is itself then that element do not appear in a cycle.

eg. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$

then in cycle $f = (1\,2\,5\,4)$

it means $f(1) = 2$, $f(2) = 5$, $f(5) = 4$, $f(4) = 1$

Since 3 is not in the cycle so $f(3) = 3$

The identity ~~trans~~ permutation is written as $(1)$

—x—