# The Greatest Integer Function

Broafer Das

**Def:-** For a real number $x$, the greatest integer of $x$ denoted by $[x]$ is the greatest integer less than or equal to $x$.

It can be treated as a function from $\mathbb{R}$ to $\mathbb{Z}$. This function is called the greatest integer function.

Now we want to see how many times a p[rime] $p$ appears in $n!$

For example (i) $p = 5$. $n = 15$ then

$$15! = \underset{*}{15}.14.13.12.11.\underset{*}{10}.9.8.7.6.\underset{*}{5}.4.3.2.1$$

5 appears in the number marked as $*$, i.e. 5 appears 3 times. as $\frac{15}{5} = 3$

(ii) $p = 3$. and $n = 13$ then

$$13! = 13.\underset{*}{12}.11.10.\underset{*}{9}.8.7.\underset{*}{6}.5.4.\underset{*}{3}.2.1$$

3 appear 4 times but $\frac{13}{3} \neq 4$, but $\left[\frac{13}{3}\right] =$

So most expected result is—

If $n$ is a natural number and $p$ is a prim[e] then $p$ appears $\left[\frac{n}{p}\right]$ times in $n!$.

So the next theorem—

Theorem:- If $n$ is a positive real number, then the exponent of the highest power of $p$ that divides $n!$ is $\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$. where the series is finite because $\left[\frac{n}{p^k}\right]=0$ for $p^k > n$

Proof:- Among the integers

$1, 2, 3, \cdots n$. those are divisible by $p$ are $p, 2p, 3p, \cdots tp$ where $tp \le n$.

ie $t$ is the greatest integer less than or equal to $\frac{n}{p}$

ie $t = \left[\frac{n}{p}\right]$

Similarly among $1, 2, 3, \cdots n$ ;; $p^2, 2p^2, 3p^2, \cdots \left[\frac{n}{p^2}\right]p^2$ are the integers divisible by $p^2$

Similarly $p^k, 2p^k, \cdots \left[\frac{n}{p^k}\right]p^k$ are the integers divisible by $p^k$.

Thus ~~to total~~ total number of times $p$ divides $n!$ is $\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]$

For example:- ① Let $n=5$ and $p=3$. then

$$\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right] = \left[\frac{5}{3}\right]+\left[\frac{5}{3^2}\right] = 1+0 = 1$$

Now $5! = 5.4.3.2.1$. clearly $3 | 5!$ but $3^2 \nmid 5!$ ie the height power is 1.

(ii) $n = 10!$ and $n = 3$ then the height pawer of 3 dalrich devider 10! is

$$\left[\frac{10}{3}\right] + \left[\frac{10}{3^2}\right] + \left[\frac{10}{3^3}\right]$$

$$= 3 + 1 + 0 = 4$$

Theorem:- If $n$ and $r$ are pasitive integer which $1 \leq r < n$ the binomial co-efficient ${}^{n}C_{r}$ is an integer.

Pf.- If $a$ and $b$ are arbitrary real member then

$[a+b] \geq [a] + [b]$

Thus for each prime factor $p$ of $r!(n-r)!$

$$\left[\frac{n}{p^k}\right] \geq \left[\frac{r}{p^k}\right] + \left[\frac{n-r}{p^k}\right], \quad k = 1, 2, ---$$

Taking sum for all k's we get

$$\sum_{k \geq 1} \left[\frac{n}{p^k}\right] \geq \sum_{k \geq 1} \left[\frac{r}{p^k}\right] + \sum_{k \geq 1} \left[\frac{n-r}{p^k}\right] \quad \text{——— (1)}$$

The L.H.S of (1) is the heighest power of $p$ that devider $n!$

The $\sum_{k \geq 1} \left[\frac{r}{p^k}\right]$ is the heighest power of $p$ that devider $r!$

$\sum_{k \geq 1} \left[\frac{n-r}{p^k}\right]$ is the heeighest power of $p$ that devide $(n-r)!$

i.e The heighest power of $p$ that devider $n!$ is greather than equal to the sum of height power of $p$ that devider $r!$ and $(n-r)!$. So ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is an integer.

Cor:- The product of any $r$ consecutive integers is divisible by $r!$

Pf:- Let $n(n-1)---(n-r+1)$ be any product of $r$ consecutive integers then

$$n(n-1)---(n-r+1) = \frac{n!}{(n-r)!}$$

$$= \left(\frac{n!}{r!(n-r)!}\right) r!$$

$$= {}^{n}C_{r} \cdot r!$$

$$\text{ie } r! \mid n(n-1)---(n-r+1)$$

Theorem:- Let $f$ and $F$ be number-theoretic functions such that $F(n) = \sum_{d \mid n} f(d)$. Then for any positive integer $N$,

$$\sum_{n=1}^{N} F(n) = \sum_{k=1}^{N} f(k) \left[\frac{N}{k}\right]$$

(Proof of this theorem we will do in class)

Cor:- If $N$ is a positive integer then $\sum_{n=1}^{N} \tau(n) = \sum_{n=1}^{N} \left[\frac{N}{n}\right]$

Pf:- We know that $\tau(n) = \sum_{d \mid n} 1$. Then by the theorem

$$\sum_{n=1}^{N} \tau(n) = \sum_{k=1}^{N} 1 \cdot \left[\frac{N}{k}\right]$$. putting $n$ instead of $k$ we get

$$\sum_{n=1}^{N} \tau(n) = \sum_{n=1}^{N} \left[\frac{N}{n}\right]$$

Similarly

$$\sum_{n=1}^{N} \sigma(n) = \sum_{n=1}^{N} n \left[\frac{N}{n}\right]$$